

ON THE ENUMERATION OF ROOTED TREES WITH FIXED SIZE OF MAXIMAL DECREASING TREES

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ABSTRACT. Let \mathcal{T}_n be the set of rooted labeled trees on $\{0, \dots, n\}$. A maximal decreasing subtree of a rooted labeled tree is defined by the maximal subtree from the root with all edges being decreasing. In this paper, we study a new refinement $\mathcal{T}_{n,k}$ of \mathcal{T}_n , which is the set of rooted labeled trees whose maximal decreasing subtree has $k + 1$ vertices.

1. INTRODUCTION

For a nonnegative integer n , let \mathcal{T}_n be the set of rooted labeled trees on $[0, n] := \{0, \dots, n\}$. For a given rooted labeled tree T , a *maximal decreasing subtree* of T is defined by the maximal subtree from the root with all edges being decreasing, denoted by $\text{MD}(T)$. Figure 1 illustrates the maximal decreasing subtree of a given tree T . Let $\mathcal{T}_{n,k}$ be the set of rooted labeled trees in \mathcal{T}_n whose maximal decreasing subtree has $k + 1$ vertices.

Within the scope of proven research, a maximal decreasing subtree first appeared in the paper [CDG00] of Chauve, Dulucq, and Guibert, for constructing the bijection between $\mathcal{T}_{n,0}$ and the set of trees in \mathcal{T}_n with n being a leaf. Recently, Bergeron and Livernet [BL10] introduced it in order to analyze the free Lie algebra based on rooted labeled trees. None of them mentioned, however, the refined set $\mathcal{T}_{n,k}$ nor considered the enumeration of $\mathcal{T}_{n,k}$.

In Section 2, we shall count the number of elements in $\mathcal{T}_{n,k}$. We shall also introduce a set of certain functions on $[n]$, which is equinumerous to $\mathcal{T}_{n,k}$. In Section 3, we shall decompose a rooted labeled tree into rooted subtrees, each maximal decreasing subtree of which is a single vertex. Then some formulae related to $|\mathcal{T}_{n,k}|$ are given from this decomposition. In Section 4, using the inverse of the matrix $\left[\binom{i+j}{j} \right]_{0 \leq i, j \leq n}$, $\mathcal{T}_{n,k}$ can be expressed as a linear combination of $\{(n+1)^n, (n+2)^n, \dots, (2n+1)^n\}$. In the last section, we discuss bijective proofs of our results.

2. MAIN RESULTS

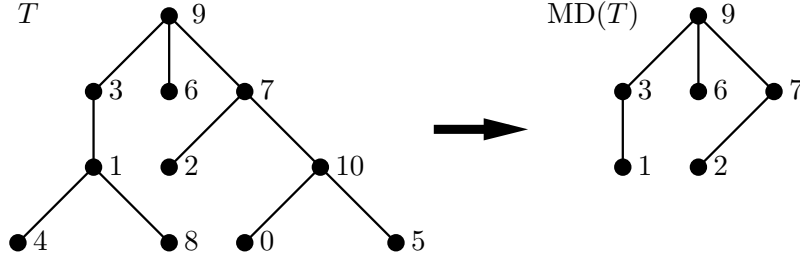
First of all, let us count the number of elements in the set $\mathcal{T}_{n,k}$.

Theorem 1. *For nonnegative integers n and k , we have*

$$|\mathcal{T}_{n,k}| = \sum_{m=k}^n \binom{n+1}{m+1} S(m+1, k+1) k! (n-k)^{n-m-1} (m-k),$$

where $S(n, k)$ is a Stirling number of the second kind.

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FIGURE 1. The maximal decreasing subtree of T from the root 9

Proof. Given a rooted labeled tree T , let V_1 be the union of the set of vertices in $\text{MD}(T)$ and the set of children of any vertex in $\text{MD}(T)$. Now, we will count the number of rooted labeled trees $T \in \mathcal{T}_{n,k}$ with $|V_1| = m + 1$.

First of all, the number of ways for selecting V_1 is equal to $\binom{n+1}{m+1}$. Make a partition of V_1 into $k + 1$ blocks, namely, B_1, \dots, B_{k+1} . The number of such partitions is equal to $S(m + 1, k + 1)$. Take the set V_0 consisting of the minimum m_i of each block B_i . Make a decreasing subtree on $V_0 = \{m_1, \dots, m_{k+1}\}$. Since it is well known that the number of (unordered) increasing trees on $k + 1$ nodes is $k!$, there are exactly $k!$ ways of making a decreasing subtree on V_0 . Append vertices in $V_1 \setminus V_0$ to this decreasing subtree such that elements in $B_i \setminus \{m_i\}$ are children of m_i for $i = 1, \dots, k + 1$. It is well-known that the number of forests F on $[0, n] \setminus V_0$ such that $V_1 \setminus V_0$ is the set of all roots of F is equal to $(n - k)^{n-m-1}(m - k)$ (see [Sta99, Prop. 5.3.2]). Since the range of m is $k \leq m \leq n$,

$$|\mathcal{T}_{n,k}| = \sum_{m=k}^n \binom{n+1}{m+1} S(m+1, k+1) k! (n-k)^{n-m-1} (m-k).$$

□

For a nonnegative integer n , let \mathcal{F}_n be the set of functions from $[n]$ to $[n]$, where $[n] := \{1, \dots, n\}$ if n is positive integer and $[0] := \emptyset$. Let $\mathcal{F}_{n,k}$ be the set of functions $f \in \mathcal{F}_n$ with $[k] \subset f([n])$, where $f([n])$ is the image of f .

Proposition 2. For nonnegative integers n and k , we have

$$|\mathcal{F}_{n,k}| = \sum_{m=k}^n \binom{n}{m} S(m, k) k! (n-k)^{n-m} \quad (1)$$

$$= \sum_{i \geq 0} (-1)^i \binom{k}{i} (n-i)^n. \quad (2)$$

Proof. Let $f^{-1}([k]) = A$ and $|A| = m$. The number of subsets A of $[n]$ of size m is equal to $\binom{n}{m}$. The function f can be decomposed into a surjection from A to $[k]$ with $S(m, k) k!$ ways and a function from $[n] \setminus A$ to $[n] \setminus [k]$ with $(n-k)^{n-m}$ ways. Since m runs through from k to n , the formula (1) holds.

Meanwhile, defining A_j by the set $\{f \in \mathcal{F}_n \mid f^{-1}(j) = \emptyset\}$, we have

$$\mathcal{F}_{n,k} = \mathcal{F}_n \setminus (A_1 \cup \cdots \cup A_k).$$

By the principle of inclusion and exclusion, we have

$$|\mathcal{F}_{n,k}| = |\mathcal{F}_n| - |A_1 \cup \cdots \cup A_k| = \sum_{I \subset [k]} (-1)^{|I|} (n - |I|)^n.$$

So, the formula (2) holds. □

Theorem 3. For nonnegative integers n and k , we have $|\mathcal{T}_{n,k}| = |\mathcal{F}_{n,k}|$, i.e.,

$$\sum_{m=k}^n \binom{n+1}{m+1} S(m+1, k+1) k! (n-k)^{n-m-1} (m-k) = \sum_{i=0}^k (-1)^i \binom{k}{i} (n-i)^n. \quad (3)$$

Proof. Since $S(m+1, k+1)(k+1)!$ is the number of surjective functions from $[m+1]$ to $[k+1]$, which is equal to $\sum_{j \geq 0} (-1)^j \binom{k+1}{j} (k+1-j)^{m+1}$ by the principle of inclusion and exclusion, it follows that

$$\begin{aligned} |\mathcal{T}_{n,k}| &= \sum_m \binom{n+1}{m+1} [S(m+1, k+1) k!] (n-k)^{n-m-1} (m-k) \\ &= \sum_m \binom{n+1}{m+1} \left[\frac{1}{k+1} \sum_{j \geq 0} (-1)^j \binom{k+1}{j} (k+1-j)^{m+1} \right] (n-k)^{n-m-1} (m-k). \end{aligned}$$

Separating the term $m-k$ to $(n-k) - (n-m)$ and changing the order of summations, we get

$$\begin{aligned} |\mathcal{T}_{n,k}| &= \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} \sum_m \binom{n+1}{m+1} (k+1-j)^{m+1} (n-k)^{n-m} \\ &\quad - \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} \sum_m (n+1) \binom{n}{m+1} (k+1-j)^{m+1} (n-k)^{n-m-1}. \end{aligned}$$

By the binomial theorem,

$$\begin{aligned} |\mathcal{T}_{n,k}| &= \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} [(n+1-j)^{n+1} - (n+1)(n+1-j)^n] \\ &= \sum_j \frac{(-1)^j}{k+1} \binom{k+1}{j} (-j)(n+1-j)^n. \end{aligned}$$

Substituting $j = i+1$ in the previous equation, the formula (3) holds. □

3. PROPERTIES

A rooted labeled tree T is called a *local minimum tree*, if $\text{MD}(T)$ consists of a single vertex. Note that $\mathcal{T}_{n,0}$ is the set of local minimum trees on $[0, n]$ and $|\mathcal{T}_{n,0}|$ is equal to n^n [CDG00]. Also $\mathcal{T}_{n,n}$ is the set of decreasing trees on $[0, n]$ and $|\mathcal{T}_{n,n}|$ is equal to $n!$.

Given $T \in \mathcal{T}_{n,k}$, we can decompose T into $k+1$ local minimum trees by removing k edges in $\text{MD}(T)$. This decomposition yields the following lemma.

Lemma 4. *For nonnegative integers n and k , we have*

$$|\mathcal{T}_{n,k}| = \frac{1}{k+1} \sum_{n_1+\dots+n_{k+1}=n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \dots n_{k+1}^{n_{k+1}}$$

Proof. It is enough to show the following formula

$$|\mathcal{T}_{n,k}| = k! \left(\frac{1}{(k+1)!} \sum_{n_1+\dots+n_{k+1}=n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \dots |\mathcal{T}_{n_{k+1},0}| \right). \quad (4)$$

First of all, we will make a tuple (T_1, \dots, T_{k+1}) of local minimum trees satisfying two conditions:

- (i) The tuple $(L(T_1), \dots, L(T_{k+1}))$ is an ordered partition of the set $[0, n]$ and
- (ii) $\text{root}(T_1) < \text{root}(T_2) < \dots < \text{root}(T_{k+1})$,

where $L(T)$ means a set of labels of vertices in T and $\text{root}(T)$ a label of the root of T .

Consider a tuple (S_1, \dots, S_{k+1}) of local minimum trees with the only condition (i). For a given sequence n_1, \dots, n_{k+1} of nonnegative integers with $n_1 + \dots + n_{k+1} = n - k$, the number of tuples (S_1, \dots, S_{k+1}) with $|L(S_i)| = n_i + 1$ is equal to

$$\binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \dots |\mathcal{T}_{n_{k+1},0}|.$$

So the number of all tuples (S_1, \dots, S_{k+1}) with the condition (i) is equal to

$$\sum_{n_1+\dots+n_{k+1}=n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \dots |\mathcal{T}_{n_{k+1},0}|.$$

From the condition (ii), the number of all tuples (T_1, \dots, T_{k+1}) is equal to

$$\frac{1}{(k+1)!} \sum_{n_1+\dots+n_{k+1}=n-k} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} |\mathcal{T}_{n_1,0}| \dots |\mathcal{T}_{n_{k+1},0}|.$$

Since the number of decreasing subtrees on $\{\text{root}(T_1), \dots, \text{root}(T_{k+1})\}$ is $k!$, we get the formula (4). \square

From Lemma 4, we can deduce the following result.

Theorem 5. *We have three exponential generating functions:*

$$1 + \sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| \frac{t^{k+1}}{k!} \frac{x^{n+1}}{(n+1)!} = \exp \left(t \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right), \quad (5)$$

$$1 + \sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| (k+1) t^{k+1} \frac{x^{n+1}}{(n+1)!} = \left(1 - t \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right)^{-1}, \quad (6)$$

$$\sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| t^{k+1} \frac{x^{n+1}}{(n+1)!} = -\ln \left(1 - t \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right). \quad (7)$$

Proof. From Lemma 4, left-hand side of three formulas become

$$1 + \sum_{n \geq 0} \sum_{k=0}^n \sum_{n_i} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \dots n_{k+1}^{n_{k+1}} \frac{t^{k+1}}{(k+1)!} \frac{x^{n+1}}{(n+1)!}, \quad (8)$$

$$1 + \sum_{n \geq 0} \sum_{k=0}^n \sum_{n_i} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \dots n_{k+1}^{n_{k+1}} t^{k+1} \frac{x^{n+1}}{(n+1)!}, \quad (9)$$

$$\sum_{n \geq 0} \sum_{k=0}^n \sum_{n_i} \binom{n+1}{n_1+1, \dots, n_{k+1}+1} n_1^{n_1} \dots n_{k+1}^{n_{k+1}} \frac{t^{k+1}}{k+1} \frac{x^{n+1}}{(n+1)!}, \quad (10)$$

where n_i means $n_1 + \dots + n_{k+1} = n - k$. Using the compositional formula for exponential structures [Sta99, Theorem 5.5.4], three formulas are of form $F(tG(x))$ where

$$G(x) = \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!}.$$

In case (8), the corresponding $F(x)$ is given by

$$F(x) = 1 + \sum_{i \geq 0} \frac{(i+1)!}{(i+1)!} \frac{x^{i+1}}{(i+1)!} = \exp(x).$$

In case (9), the corresponding $F(x)$ is given by

$$F(x) = 1 + \sum_{i \geq 0} \frac{(i+1)!}{1} \frac{x^{i+1}}{(i+1)!} = \frac{1}{1-x}.$$

In case (10), the corresponding $F(x)$ is given by

$$F(x) = \sum_{i \geq 0} \frac{(i+1)!}{(i+1)} \frac{x^{i+1}}{(i+1)!} = \ln \frac{1}{1-x}.$$

These complete the proof. □

By definition of $\mathcal{T}_{n,k}$, we have

$$\sum_{k=0}^n |\mathcal{T}_{n,k}| = (n+1)^n, \quad (11)$$

which can be also induced from $t = 1$ in (7). Similarly, putting $t = 1$ in (6) and applying the equation (5.67) in [Sta99], we get

$$\begin{aligned} 1 + \sum_{n \geq 0} \sum_{k=0}^n |\mathcal{T}_{n,k}| (k+1) \frac{x^{n+1}}{(n+1)!} &= \left(1 - \sum_{n \geq 0} n^n \frac{x^{n+1}}{(n+1)!} \right)^{-1} \\ &= 1 + \sum_{n \geq 0} (n+2)^n \frac{x^{n+1}}{(n+1)!}. \end{aligned}$$

Thus we have

$$\sum_{k=0}^n (k+1) |\mathcal{T}_{n,k}| = (n+2)^n. \quad (12)$$

From (11) and (12), we are able to deduce the followings.

Theorem 6. *For a nonnegative integers n , k , and α , we have*

$$\sum_{k=0}^n \binom{k+\alpha}{\alpha} |\mathcal{T}_{n,k}| = (n+1+\alpha)^n. \quad (13)$$

Proof. Since we have proved $|\mathcal{T}_{n,k}| = |\mathcal{F}_{n,k}|$ in Theorem 3, it is enough to show

$$\sum_{k=0}^n \binom{k+\alpha}{\alpha} |\mathcal{F}_{n,k}| = (n+1+\alpha)^n. \quad (14)$$

For $\alpha = 0$, let $\mathcal{G}_{n,k}$ be the set of functions g from $[n]$ to $[0, n]$ with $[0, k-1] \subset g([n])$ but $k \notin g([n])$. By definition of $\mathcal{G}_{n,k}$,

$$\sum_{k=0}^n |\mathcal{G}_{n,k}| = (n+1)^n.$$

There is a simple bijection φ from $\mathcal{F}_{n,k}$ to $\mathcal{G}_{n,k}$ as follows: Given a $f \in \mathcal{F}_{n,k}$, consider a function g from $[n]$ to $[0, n]$ defined by

$$g(i) = \begin{cases} f(i) - 1 & \text{if } f(i) \leq k \\ f(i) & \text{otherwise.} \end{cases}$$

Since the images of g includes $0, \dots, k-1$ but does not include k , the function g belongs to $\mathcal{G}_{n,k}$ and $\varphi(f) = g$ is well-defined. Since φ is reversible, it is a bijection. So it holds that

$$|\mathcal{F}_{n,k}| = |\mathcal{G}_{n,k}|$$

for all $0 \leq k \leq n$.

For $\alpha > 0$, let $\mathcal{H}_{n,k,\alpha}$ be the set of functions g from $[n]$ to $[-\alpha, n]$ with

$$|[-\alpha, k-1] \setminus g([n])| = \alpha$$

and $k \notin g([n])$, where $[a, b] := \{a, \dots, b\}$. For every function h from $[n]$ to $[-\alpha, n]$, since the cardinality of the domain is less than the cardinality of the codomain by $\alpha + 1$, there exists a unique k satisfying above conditions. Note that k is the $(\alpha + 1)$ -st element in $[-\alpha, n] \setminus g([n])$. Thus

$$\sum_{k=0}^n |\mathcal{H}_{n,k,\alpha}| = (n + 1 + \alpha)^n.$$

Let \mathcal{A} be the set of all α -elements subsets of $[-\alpha, k-1]$. Clearly, $|\mathcal{A}| = \binom{k+\alpha}{\alpha}$. There is a bijection from $\mathcal{A} \times \mathcal{F}_{n,k}$ to $\mathcal{H}_{n,k,\alpha}$ as follows: For a given $(A, f) \in \mathcal{A} \times \mathcal{F}_{n,k}$, we make a $(A, \varphi(f)) \in \mathcal{A} \times \mathcal{G}_{n,k}$. Consider the order-preserving bijection σ from $[0, n]$ to $[-\alpha, n] \setminus A$. Then we can define the function h from $[n]$ to $[-\alpha, n]$ by

$$h = \sigma \circ (\varphi(f))$$

and this function h is contained in $\mathcal{H}_{n,k,\alpha}$. Hence,

$$|\mathcal{H}_{n,k,\alpha}| = |\mathcal{A} \times \mathcal{F}_{n,k}| = |\mathcal{A}| \cdot |\mathcal{F}_{n,k}| = \binom{k+\alpha}{\alpha} |\mathcal{F}_{n,k}|$$

for all $0 \leq k \leq n$. □

For example, let $n = 5$, $k = 2$, and $\alpha = 3$. Take $A = \{-2, -1, 1\} \in \binom{[-3,1]}{3}$ and $f = (f(1), \dots, f(5)) = (5, 2, 1, 3, 2) \in \mathcal{F}_{5,2}$. Then $g = \varphi(f) \in \mathcal{G}_{5,2}$ and $h = \sigma \circ (\varphi(f)) \in \mathcal{H}_{5,2,3}$ are given by

$$\begin{aligned} (g(1), \dots, g(5)) &= (5, 1, 0, 3, 1), \\ (h(1), \dots, h(5)) &= (5, 0, -3, 3, 0). \end{aligned}$$

Let us consider the equation (13) or (14) for a negative integer α . In fact, the left hand sides of these equations are not well-defined even for $\alpha = -1$, nevertheless the right hand sides are. Here we find, however, the coefficients of $|\mathcal{T}_{n,k}|$ that can replace the term $\binom{k-1}{-1}$ as follows.

Theorem 7. *For a positive integer n , we have*

$$\sum_{k=1}^n \frac{1}{k} |\mathcal{T}_{n,k}| = \sum_{k=1}^n \frac{1}{k} |\mathcal{F}_{n,k}| = n^n. \quad (15)$$

Proof. From Proposition 2 and Theorem 3, expanding $(n - i)^n$ by the binomial theorem, we have

$$\begin{aligned} |\mathcal{T}_{n,k}| &= |\mathcal{F}_{n,k}| = \sum_{i \geq 0} (-1)^i \binom{k}{i} (n - i)^n \\ &= \sum_{i \geq 0} (-1)^i \binom{k}{i} \sum_{j \geq 0} \binom{n}{j} n^{n-j} (-i)^j. \end{aligned}$$

Hence the left hand side of (15) is

$$\sum_{k=1}^n \frac{1}{k} |\mathcal{T}_{n,k}| = \sum_{j \geq 0} \sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} \binom{n}{j} n^{n-j} (-i)^j.$$

We divide it into three cases; $j = 0$, $j = 1$, and $j > 1$.

In case of $j = 0$,

$$\sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} n^n = n^n \sum_{k=1}^n \frac{1}{k} (1-1)^k = 0. \quad (16)$$

In case of $j = 1$,

$$\begin{aligned} \sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} n^n (-i) &= n^n \sum_{k=1}^n \sum_{i \geq 0} (-1)^{i-1} \binom{k-1}{i-1} \\ &= n^n \sum_{k=1}^n (1-1)^{k-1} = n^n. \end{aligned} \quad (17)$$

In case of $j > 1$,

$$\begin{aligned} \sum_{j > 1} \sum_{k=1}^n \sum_{i \geq 0} \frac{1}{k} (-1)^i \binom{k}{i} \binom{n}{j} n^{n-j} (-i)^j &= \sum_{j > 1} \sum_{i \geq 0} (-1)^i \binom{n}{j} n^{n-j} (-1)^j i^{j-1} \sum_{k=1}^n \binom{k-1}{i-1} \\ &= \sum_{j > 1} \sum_{i \geq 0} (-1)^{i+j} \binom{n}{j} n^{n-j} i^{j-1} \binom{n}{i} \\ &= \sum_{j > 1} (-1)^j \binom{n}{j} n^{n-j} \left[\sum_{i \geq 0} (-1)^i \binom{n}{i} i^{j-1} \right] = 0. \end{aligned} \quad (18)$$

Note that, by the principle of inclusion and exclusion, the expression $\sum_{i \geq 0} (-1)^i \binom{n}{i} i^{j-1}$ in (18) is the number of surjections from $[j-1]$ to $[n]$. Since $j-1 < n$, it is zero.

From (16), (17), and (18), we finally obtain $\sum_{k=1}^n \frac{1}{k} |\mathcal{T}_{n,k}| = 0 + n^n + 0 = n^n$. \square

4. INVERSE RELATION

For nonnegative integer n , let $A(n)$ be the square matrix of size $n+1$ defined by

$$A(n) := \left[\binom{i+j}{i} \right]_{0 \leq i, j \leq n}.$$

Define two column vectors $t(n)$ and $p(n)$ by

$$t(n) := \begin{pmatrix} |\mathcal{T}_{n,0}| \\ |\mathcal{T}_{n,1}| \\ \vdots \\ |\mathcal{T}_{n,n}| \end{pmatrix} \quad \text{and} \quad p(n) := \begin{pmatrix} (n+1)^n \\ (n+2)^n \\ \vdots \\ (2n+1)^n \end{pmatrix}.$$

Then the equation (13) can be interpreted as

$$A(n) t(n) = p(n).$$

The matrix $A(n)$ is nonsingular. Moreover, we can compute its inverse directly. Let $B(n)$ be the square matrix of size $n+1$ defined by

$$B(n) := \left[(-1)^{i+j} \sum_{m=0}^n \binom{m}{i} \binom{m}{j} \right]_{0 \leq i, j \leq n}.$$

Theorem 8. *For a nonnegative integer n , the two matrices $A(n)$ and $B(n)$ are inverse matrices of each other.*

Proof. Here is the calculation:

$$\begin{aligned} \sum_{l=0}^n A(n)_{i,l} B(n)_{l,j} &= \sum_{l=0}^n \binom{i+l}{i} (-1)^{l+j} \sum_{m=0}^n \binom{m}{l} \binom{m}{j} \\ &= \sum_{m=0}^n (-1)^{m-j} \binom{m}{j} \left[\sum_{l=0}^n \binom{i+l}{l} \binom{m}{m-l} (-1)^{m-l} \right] \end{aligned}$$

Comparing the coefficients of q^m for $(1-q)^{-(i+1)}(1-q)^m = (1-q)^{-(i+1-m)}$, we obtain

$$\sum_{l=0}^n \binom{i+l}{l} \binom{m}{m-l} (-1)^{m-l} = \binom{i}{m}.$$

Thus, we have

$$\begin{aligned} \sum_{l=0}^n A(n)_{i,l} B(n)_{l,j} &= \sum_{m=0}^n (-1)^{m-j} \binom{m}{j} \binom{i}{m} \\ &= \binom{i}{j} \sum_{m=0}^n (-1)^{m-j} \binom{i-j}{m-j} \binom{i}{j} (1-1)^{i-j} = \delta_{i,j} \end{aligned}$$

which completes the proof. \square

From the matrix identity $B(n) p(n) = t(n)$, we obtain another expression for $|\mathcal{T}_{n,k}|$.

Corollary 9. *For nonnegative integers n and k , we have*

$$|\mathcal{T}_{n,k}| = \sum_{0 \leq l \leq m \leq n} (-1)^{k+l} \binom{m}{k} \binom{m}{l} (n+1+l)^n. \quad (19)$$

5. REMARKS

Since $|\mathcal{T}_{n,k}| = |\mathcal{F}_{n,k}|$, it is desired to construct a bijection between $\mathcal{T}_{n,k}$ and $\mathcal{F}_{n,k}$ for all $0 \leq k \leq n$. Also, it is natural to ask a bijective proof of (15). Recently, Jang Soo Kim [Kim11] constructed bijections for the above questions. It would be interesting to give a combinatorial explanation of (19).

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